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1998 J. Phys. A: Math. Gen. 31 7403

(http://iopscience.iop.org/0305-4470/31/36/011)

J. Phys. A: Math. Gen. **31** (1998) 7403–7413. Printed in the UK PII: S0305-4470(98)92746-9

Lagrangian and Hamiltonian constraint structure coefficients

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Received 25 March 1998, in final form 6 July 1998

Abstract. We introduce the notion of Lagrangian and Hamiltonian constraint structure coefficients. These coefficients appear in the process of investigating the consistency condition on constraints in both formulations. Then we show that for a first-class Hamiltonian constrained system one can find all the Lagrangian coefficients in terms of Hamiltonian ones.

1. Introduction

The study of constrained systems has traditionally been pursued within the Hamiltonian formulation [1]. However, the Lagrangian approach, which is more direct but more difficult to handle, has also been considered [2, 8].

The question which naturally arises is whether there is a one-to-one correspondence between the two approaches. Several authors have studied the problem of the equivalence of the Lagrangian and Hamiltonian formulations of the constrained systems. For example, a complete and exact analysis can be found in [3, 4]. There it was shown that the Hamiltonian constraints at each level can be divided into first- and second-class constraints. These are weakly related to projectable and non-projectable Lagrangian constraints respectively, via the action of time evolution operator. It was also shown that projectable Lagrangian constraints are weakly equal to the pull-back of the first-class Hamiltonian constraints (see relation (41) below). These two sets of constraints can be used in a complicated way to write down the generator of gauge transformation in each formulation. See [5, 6] for the Hamiltonian and [7, 8] for the Lagrangian formulations.

However, in some contexts, it is not just enough to know weak relations between constraints. For example, in constructing the BRST generator, the strong algebra of the Hamiltonian constraints is required [9]. Similarly, one needs the exact structure of Hamiltonian or Lagrangian constrained systems (in the form of strong relations between the constraints) in order to construct the generator of gauge transformations in both approaches [5–8].

In this paper we analyse the Lagrangian and Hamiltonian constraint structure for a system which has only first-class constraints in the Hamilton–Dirac formulation, i.e. a first-class system (section 2). Then we introduce the notion of the Lagrangian (Hamiltonian) constraint structure coefficients, hereafter abbreviated to LCSCs (HCSCs). Such a

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description of constraint structures is achieved at the cost of restricting ourselves to first-class systems. It seems that for the general case, it is much more difficult to write down strong relation between constraints in a closed form. However, the above restriction does not exclude the important class of gauge-invariant theories which are of utmost importance for physicists.

In section 3, which is the main part of this paper, we have shown that beginning with the Hamiltonian constraint structure, one can construct the Lagrangian structure step by step. In other words, given HCSCs for a first-class system one can obtain all the LCSCs and the coefficients of the relations between the two sets of constraints. In order to clarify the method, an example is presented in section 4.

For clarity, we have considered only finite-dimensional dynamical systems. Generalization to field theory seems to be straightforward. We also assume that all of the constraints are effective. In other words, the gradient of the constraints dose not vanish on the constraint surface.

2. Constraint structure coefficients

2.1. The Lagrangian formulation

Consider the action

$$S = \int \mathrm{d}t\ L(q, \dot{q}) \tag{1}$$

where the Lagrangian $L(q, \dot{q})$ is a function on TQ, the velocity phase space. The configuration space Q is finite dimensional with local coordinates $q_i (i = 1, \ldots, n)$.

For a singular Lagrangian, the Eulerian derivatives

$$L_i \equiv -\frac{\delta S}{\delta q_i(t)} = W_{ij}\ddot{q}_j - \alpha_i \qquad i = 1, \ldots, n \tag{2}$$

where

$$W_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \tag{3}$$

and

$$\alpha_i = \frac{\partial L}{\partial q_i} - \dot{q}_k \frac{\partial^2 L}{\partial q_k \partial \dot{q}_i} \tag{4}$$

are not independent functions of accelerations. This means that the Hessian matrix W is singular and has some null-eigenvectors $\gamma^\mu(q, \dot{q})$:

$$\gamma_i^\mu W_{ij} = 0 \qquad \mu = 1, \ldots, m. \tag{5}$$

Multiplying equations (2) by the null-eigenvectors (5) gives the primary Lagrangian constraints

$$\chi_\mu^{(1)}(q, \dot{q}) = \gamma_i^\mu(q, \dot{q})\alpha_i(q, \dot{q}) \qquad \mu = 1, \ldots, m. \tag{6}$$

The consistency conditions on these constraints require that

$$\frac{\mathrm{d}\chi_\mu^{(1)}}{\mathrm{d}t} = \frac{\partial \chi_\mu^{(1)}}{\partial q_i}\dot{q}_i + \frac{\partial \chi_\mu^{(1)}}{\partial \dot{q}_i}\ddot{q}_i \tag{7}$$

vanishes, which should be considered along with the Euler–Lagrange equations of motion $L_i = 0$. We assume that no new equation for the accelerations would emerge from the consistency conditions. It was shown in [3] that this condition restricts the Lagrangian constraints to projectable ones and the Hamiltonian constraints to first-class ones. Therefore the acceleration term in (7) can at most be a linear combination of the L_i. The consistency condition (7) can be written as

$$\frac{\mathrm{d}\chi_\mu^{(1)}}{\mathrm{d}t} = \chi_\mu^{(2)} + b_{\mu i}^{(1)}(q,\dot{q})L_i(q,\dot{q},\ddot{q}) \tag{8}$$

where the $\chi_\mu^{(2)}$ are recognized as second-level Lagrangian constraints. We should proceed iteratively, and at each step use the consistency conditions on the secondary Lagrangian constraints. However, in the higher steps, the acceleration term in $\mathrm{d}\chi_\mu^{(l)}/\mathrm{d}t$ can itself be a weakly vanishing term containing the previous constraints, as follows:

$$\frac{\mathrm{d}\chi_\mu^{(s)}}{\mathrm{d}t} = \chi_\mu^{(s+1)} + b_{\mu i}^{(s)}(q,\dot{q})L_i + \sum_{t=1}^{s-1} h_{\mu\nu j}^{(s,t)}(q,\dot{q})\chi_\nu^{(t)}\ddot{q}_j \qquad s = 2,\ldots,k-1. \tag{9}$$

It should be noted that the choice of $b_{\mu i}^{(s)}$ and $\chi_\mu^{(s+1)}$ is not unique. In fact, they can be redefined according to

$$\begin{aligned} b_{\mu i}^{(s)} &\to b_{\mu i}^{'(s)} = b_{\mu i}^{(s)} + \Lambda_{\mu\nu}\gamma_i^\nu \\ \chi_\mu^{(s+1)} &\to \chi_\mu^{'(s+1)} = \chi_\mu^{(s+1)} + \Lambda_{\mu\nu}\alpha_i\gamma_i^\nu. \end{aligned} \tag{10}$$

For a system with a finite number of degrees of freedom (or with a finite number of fields in a field theory), the algorithm for consistency of the constraints should stop at some level k. At the last level, $\mathrm{d}\chi_\mu^{(k)}/\mathrm{d}t$ should vanish weakly on the shell $L_i = 0$, i.e.

$$\frac{\mathrm{d}\chi_\mu^{(k)}}{\mathrm{d}t} = \sum_{l=1}^{k} a_{\mu\nu}^{(l)}(q,\dot{q})\chi_\nu^{(l)} + b_{\mu i}^{(k)}(q,\dot{q})L_i + \sum_{t=1}^{k-1} h_{\mu\nu j}^{(k,t)}(q,\dot{q})\chi_\nu^{(t)}\ddot{q}_j. \tag{11}$$

Relations (8)–(11) give the Lagrangian structure of a constraint system, as pointed out in [7]. We call the set of coefficients a, b and h, which are functions of $(q,\dot{q})$, the *Lagrangian constraint structure coefficients* (LCSCs).

It was shown [7, 8] that relations (11) lead to Noether identities which are the necessary and sufficient conditions for the invariance of the action under gauge transformation. Under the assumption of projectibility of the Lagrangian constraints, the number of the Noether identities and/or the independent functions of time in the gauge transformation is equal to m, the number of primary constraints (6). Each primary constraint, indexed by μ, is at the top of a chain of constraints. We have assumed that the length of the chains (k) is the same for all μ's. In the general case when one relaxes this simplifying assumption, a more detailed analysis is required.

2.2. *The Hamiltonian formulation*

Consider the Legendre transformation (called the FL map) from $(q,\dot{q})$ to (q,p), defined within the relations

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \equiv \mathcal{P}_i(q,\dot{q}) \qquad i = 1,\ldots,n. \tag{12}$$

For a singular Lagrangian, the FL map is singular. This is because the p_i are not independent functions of the velocities and the determinant of $W_{ij} = \partial\mathcal{P}_i/\partial\dot{q}_j$ vanishes. So under FL, the whole space TQ would be mapped onto a subspace M_0 of the phase space, defined by the vanishing of certain functions

$$\Phi_\mu^{(0)}(q,p) \qquad \mu = 1,\ldots,m \tag{13}$$

called primary (Hamiltonian) constraints. Their number is the same as the dimension of the null-eigenspace of the Hessian matrix W. We define the operator FL^* on any function of phase space as

$$\mathrm{FL}^*\, g(q, p) \equiv g(q, \mathcal{P}(q, \dot{q})). \tag{14}$$

The inverse operator is not well defined, since the map (12) is not invertible for all the $\dot{q}_i$. Since the primary Hamiltonian constraints are direct consequences of the definition of momenta (12), one can write

$$\mathrm{FL}^*\, \Phi_\mu^{(0)}(q, p) = 0. \tag{15}$$

By definition, a function $g(q, \dot{q})$ is FL-projectable, if there exists a function $f(q, p)$ in the phase space such that $g(q, \dot{q}) = \mathrm{FL}^*\, f(q, p)$. It was shown in [3] that the necessary and sufficient condition for $g(q, \dot{q})$ to be FL-projectable is that

$$\Gamma_\mu g(q, \dot{q}) \equiv \gamma_\mu^i \frac{\partial}{\partial \dot{q}_i} g = 0. \tag{16}$$

The canonical Hamiltonian H can be defined as $H(q, p) = \mathrm{FL}h(q, \dot{q})$, where $h(q, \dot{q}) = \dot{q}_i \mathcal{P}_i(q, \dot{q}) - L(q, \dot{q})$ is the energy function and satisfies the condition (16).

It is well known [1] that the total Hamiltonian

$$H_{\mathrm{t}}(q, p) = H(q, p) + v_\mu \Phi_\mu^{(0)}(q, p) \tag{17}$$

is responsible of the dynamics of the system in the phase space, where the v_μ are the undetermined Lagrange multipliers. That is, if $q(t)$ is a solution of the Euler–Lagrange equations of motion and $p(t) \equiv \mathcal{P}(q(t), \mathrm{d}q/\mathrm{d}t)$, then for any function $g(q(t), p(t))$ in the phase space we have [3]

$$\frac{\mathrm{d}g}{\mathrm{d}t} \approx [g, H] + v_\mu [g, \Phi_\mu^{(0)}]. \tag{18}$$

Using this equation for the phase space coordinates of the point (q, p) and considering the inverse map $\mathrm{FL}^{-1} : (q, p) \to (q, \dot{q})$ and the Euler–Lagrange equation $\mathrm{d}\mathcal{P}_i/\mathrm{d}t = -\partial L/\partial q_i$, we can write

$$\dot{q}_i = \mathrm{FL}^* \frac{\partial H}{\partial p_i} + v_\mu(q, \dot{q})\, \mathrm{FL}^* \frac{\partial \Phi_\mu^{(0)}}{\partial p_i} \tag{19}$$

$$\frac{\partial L}{\partial q_i} = -\,\mathrm{FL}^* \frac{\partial H}{\partial q_i} - v_\mu(q, \dot{q})\, \mathrm{FL}^* \frac{\partial \Phi_\mu^{(0)}}{\partial q_i}. \tag{20}$$

Here we have noted that $\mathrm{FL}^{-1}(q, p)$ exists only for a subspace of the phase space which is the FL map of TQ. Furthermore, under FL^{-1} the point (q, p) maps to a subspace of TQ, according to the choice of the v_μ. So for a special point $(q, \dot{q})$ in TQ, the value of the v_μ depend on that point and are functions of $(q, \dot{q})$.

We are especially interested in the dynamics of the constraints. They should remain valid in the course of the time. Suppose the primary constraints are *first class*, i.e. they have weakly vanishing Poisson brackets with each other. So, equation (18) for the primary constraints $\Phi_\mu^{(0)}$ leads to secondary constraints $\Phi_\mu^{(1)}$:

$$\Phi_\mu^{(1)} = [\Phi_\mu^{(0)}, H]. \tag{21}$$

We should go further to investigate the consistency condition for secondary constraints and so on, such that the consistency condition for some level-n constraint, $\Phi_\nu^{(n)}$, leads to the recurrence relation

$$\Phi_\nu^{(n+1)} = [\Phi_\nu^{(n)}, H]. \tag{22}$$

We have assumed that the constraints at all levels are first class. This means that the consistency conditions for $\Phi_\mu^{(n)}$, does not lead to the determination of the Lagrange multipliers in terms of phase space coordinates. Therefore, the Poisson bracket of $\Phi_\mu^{(n)}$ with the primary constraints should vanish on the surface of constraints known up to that level, i.e.

$$[\Phi_\mu^{(n)}, \Phi_\nu^{(0)}] = \sum_{l=0}^{n} A_{\mu\nu\eta}^{nl}(q, p)\Phi_\eta^{(l)}. \tag{23}$$

Using the Jaccobi identity, it is not difficult to show that condition (23) suffices to treat a first-class system in the Dirac terminology. That is, the Poisson brackets of all the constraints with each other vanish on the constraint surface. However, relation (23) is the only thing that we need in what follows.

Another point to note is that one can add to the above $\Phi_\nu^{(n+1)}$ an arbitrary combination of the constraints up to level n. However, we consider the strong relation (22) as the definition of the $(n+1)$th level constraints.

The set of all Hamiltonian constraints can be classified as constraint chains. Each chain begins with a primary constraint $\Phi_\mu^{(0)}$, goes down via the relation (22) and stops at some finite level k. Suppose for simplicity that k is independent of μ, i.e. the chains are of the same length. At the last step the consistency condition should hold identically on the surface of all constraints, as follows:

$$[\Phi_\mu^{(k)}, H] = \sum_{l=0}^{k} B_{\mu\nu}^{l}(q, p)\Phi_\nu^{(l)}. \tag{24}$$

We call the coefficients $A_{\mu\nu\eta}^{nl}(q, p)$ and $B_{\mu\nu}^{l}(q, p)$ the *Hamiltonian constraint structure coefficients* (HCSCs).

3. Relation between constraint structure coefficients

In this section we investigate the relation between Hamiltonian and Lagrangian constraints. We will also show that given the HCSCs, one can, in principle, derive all the LCSCs. First we prove a lemma, which relates $\mathrm{FL}^* \Phi_\mu^{(n)}$ to the previous Hamiltonian constraints, i.e. is

Lemma 1.

$$\mathrm{FL}^* \Phi_\mu^{(n+1)} = \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{FL}^* \Phi_\mu^{(n)} - v_\nu(q, \dot{q})\, \mathrm{FL}^*\left(\sum_{m\leqslant n} A_{\mu\nu\lambda}^{nm}\Phi_\lambda^{(m)}\right) - L_i\left(\mathrm{FL}^* \frac{\partial \Phi_\mu^{(n)}}{\partial p_i}\right). \tag{25}$$

Proof. Consider the phase-space relation (22). Applying FL^* to both sides and using (19) and (20) gives

$$\mathrm{FL}^* \Phi_\mu^{(n+1)} = \mathrm{FL}^*\left(\frac{\partial \Phi_\mu^{(n)}}{\partial q_i}\right)\left(\dot{q}_i - v_\nu\, \mathrm{FL}^* \frac{\partial \Phi_\nu^{(0)}}{\partial p_i}\right) - \mathrm{FL}^*\left(\frac{\partial \Phi_\mu^{(n)}}{\partial p_i}\right)\left(-\frac{\partial L}{\partial q_i} - v_\nu\, \mathrm{FL}^* \frac{\partial \Phi_\nu^{(0)}}{\partial q_i}\right)$$

which can be written as

$$\mathrm{FL}^* \Phi_\mu^{(n+1)} = \dot{q}_i\, \mathrm{FL}^* \frac{\partial \Phi_\mu^{(n)}}{\partial q_i} + \frac{\partial L}{\partial q_i}\, \mathrm{FL}^* \frac{\partial \Phi_\mu^{(n)}}{\partial p_i} - v_\nu\, \mathrm{FL}^*[\Phi_\mu^{(n)}, \Phi_\nu^{(0)}]. \tag{26}$$

Using the definition of Euler derivatives in relations (2)–(4) we have

$$\frac{\partial L}{\partial q_i} = -L_i + \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}\ddot{q}_j + \frac{\partial^2 L}{\partial q_j \partial \dot{q}_i}\dot{q}_j. \tag{27}$$

Inserting this in (26) and using (3) and (12) gives

$$\mathrm{FL}^*\,\Phi_\mu^{(n+1)} = \dot{q}_i\left(\mathrm{FL}^*\,\frac{\partial\Phi_\mu^{(n)}}{\partial q_i} + \frac{\partial\mathcal{P}_j}{\partial q_i}\,\mathrm{FL}^*\,\frac{\partial\Phi_\mu^{(n)}}{\partial p_i}\right) + \ddot{q}_i\left(\frac{\partial\mathcal{P}_j}{\partial\dot{q}_i}\,\mathrm{FL}^*\,\frac{\partial\Phi_\mu^{(n)}}{\partial p_j}\right)$$
$$- v_\nu\,\mathrm{FL}^*[\Phi_\mu^{(n)}, \Phi_\nu^{(0)}] - L_i\left(\mathrm{FL}^*\,\frac{\partial\Phi_\mu^{(n)}}{\partial p_i}\right)$$

which, using (23), gives the required result (25). □

As we shall show, iterative use of this lemma enables us to derive relations between the HCSCs and LCSCs at all levels. First consider the case $n = 0$. For primary constraints the first two terms in (25) vanish due to (15), leading to

$$\mathrm{FL}^*\,\Phi_\mu^{(1)} = -\left(\mathrm{FL}^*\,\frac{\partial\Phi_\mu^{(0)}}{\partial p_i}\right)L_i. \tag{28}$$

On the other hand, differentiating (15) with respect to $\dot{q}_i$ gives

$$\frac{\partial\mathcal{P}_j(q,\dot{q})}{\partial\dot{q}_i}\,\mathrm{FL}^*\,\frac{\partial\Phi_\mu^{(0)}}{\partial p_j} = 0$$

which says that $\mathrm{FL}^*\,(\partial\Phi_\mu^{(0)}/\partial p_i)$ is some null-eigenvector of the Hessian matrix. So the $\gamma_\mu^i(q,\dot{q})$ in relation (5) can be chosen such that

$$\mathrm{FL}^*\,\frac{\partial\Phi_\mu^{(0)}}{\partial p_i} = \gamma_\mu^i(q,\dot{q}). \tag{29}$$

For future use, we can also differentiate (15) with respect to q_i, which gives

$$\mathrm{FL}^*\,\frac{\partial\Phi_\mu^{(0)}}{\partial q_i} + \frac{\partial\mathcal{P}_j(q,\dot{q})}{\partial q_i}\,\mathrm{FL}^*\,\frac{\partial\Phi_\mu^{(0)}}{\partial p_j} = 0$$

which with the choice (29) yields,

$$\mathrm{FL}^*\,\frac{\partial\Phi_\mu^{(0)}}{\partial q_i} = -\gamma_\mu^j(q,\dot{q})\frac{\partial^2 L}{\partial q_i\,\partial\dot{q}_j}. \tag{30}$$

Inserting (29) into (28) and recalling (2) and (6), we have

$$\mathrm{FL}^*\,\Phi_\mu^{(1)} = \chi_\mu^{(1)}. \tag{31}$$

Then at the second step, consider $n = 1$ in (25), which by using (15) and (31) leads to

$$\mathrm{FL}^*\,\Phi_\mu^{(2)} = \frac{\mathrm{d}}{\mathrm{d}t}\chi_\mu^{(1)} - v_\nu(\mathrm{FL}^*\,A_{\mu\nu\lambda}^{11})\chi_\lambda^{(1)} - L_i\left(\mathrm{FL}^*\,\frac{\partial\Phi_\mu^{(1)}}{\partial p_i}\right). \tag{32}$$

Comparing this with the Lagrangian consistency condition (8) suggests the following identifications:

$$b_{\mu i}^{(1)} = \mathrm{FL}^*\,\frac{\partial\Phi_\mu^{(1)}}{\partial p_i} \tag{33}$$

$$\chi_\mu^{(2)} = \mathrm{FL}^*\,\Phi_\mu^{(2)} + g_{\mu\lambda}^{(2,1)}\chi_\lambda^{(1)} \tag{34}$$

where the coefficients

$$g_{\mu\lambda}^{(2,1)}(q,\dot{q}) = v_\nu(q,\dot{q})\,\mathrm{FL}^*\,A_{\mu\nu\lambda}^{11} \tag{35}$$

can be viewed as *second-level relation coefficients*. (There are no first-level relation coefficients, as can be seen from (31)). In suggesting $b_{\mu i}^{(1)}$ and $\chi_\mu^{(2)}$ according to (33) and (34) we have use the arbitrariness introduced after (9). Relation (34) can be written as a weak equation as follows:

$$\chi_\mu^{(2)} \stackrel{S_1}{=} \mathrm{FL}^* \, \Phi_\mu^{(2)} \tag{36}$$

where by equality on the surface S_1 we mean equality in the space TQ, modulo the first-level constraints $\chi_\lambda^{(1)}$.

We can repeat the same process once more. Then using the results of the previous steps, especially relations (31) and (34), the resultant relation will read

$$\begin{aligned}\mathrm{FL}^* \, \Phi_\mu^{(3)} = &-\frac{\mathrm{d}\chi_\mu^{(2)}}{\mathrm{d}t} + \left(\mathrm{FL}^* \frac{\partial \Phi_\mu^{(2)}}{\partial p_i} + g_{\mu\nu}^{(2,1)} \frac{\partial \Phi_\nu^{(1)}}{\partial p_i}\right) L_i + \frac{\partial g_{\mu\nu}^{(2,1)}}{\partial \dot{q}_i} \ddot{q}_i \, \chi_\nu^{(1)} \\ &- \left(\frac{\partial g_{\mu\eta}^{(2,1)}}{\partial q_i} \dot{q}_i + g_{\mu\nu}^{(2,1)} g_{\nu\eta}^{(2,1)} + v_\nu \, \mathrm{FL}^* \, A_{\mu\nu\eta}^{21} - v_\nu \, \mathrm{FL}^* \, A_{\mu\nu\lambda}^{22} g_{\lambda\eta}^{(2,1)}\right) \chi_\eta^{(1)} \\ &- (v_\nu \, \mathrm{FL}^* \, A_{\mu\nu\lambda}^{22} + g_{\mu\lambda}^{(2,1)}) \chi_\lambda^{(2)}.\end{aligned} \tag{37}$$

Comparing this with the consistency condition of the secondary Lagrangian constraints $\chi_\mu^{(2)}$ (relation (9) for $s = 2$), we can obtain the following results:

$$\begin{aligned} b_{\mu i}^{(2)} &= \mathrm{FL}^* \frac{\partial \Phi_\mu^{(2)}}{\partial p_i} + g_{\mu\nu}^{(2,1)} \frac{\partial \Phi_\nu^{(1)}}{\partial p_i} \\ h_{\mu\nu j}^{(2,1)} &= \frac{\partial g_{\mu\nu}^{(2,1)}}{\partial \dot{q}_i} \\ \chi_\mu^{(3)} &= \mathrm{FL}^* \, \Phi_\mu^{(3)} + g_{\mu\eta}^{(3,1)} \chi_\eta^{(1)} + g_{\mu\eta}^{(3,2)} \chi_\eta^{(2)} \end{aligned} \tag{38}$$

with

$$\begin{aligned} g_{\mu\eta}^{(3,1)} &= \frac{\partial g_{\mu\eta}^{(2,1)}}{\partial q_i} \dot{q}_i + g_{\mu\nu}^{(2,1)} g_{\nu\eta}^{(2,1)} + v_\nu \, \mathrm{FL}^* \, A_{\mu\nu\eta}^{21} - v_\nu \, \mathrm{FL}^* \, A_{\mu\nu\lambda}^{22} g_{\lambda\eta}^{(2,1)} \\ g_{\mu\eta}^{(3,2)} &= v_\nu \, \mathrm{FL}^* \, A_{\mu\nu\eta}^{22} + g_{\mu\eta}^{(2,1)}. \end{aligned} \tag{39}$$

The important point to note is that the *third-level relation coefficients* $g_{\mu\nu}^{(3,1)}$ and $g_{\mu\nu}^{(3,2)}$ can be derived in terms of the previously determined relation coefficients $g_{\mu\nu}^{(2,1)}$ and the Hamiltonian constraint structure coefficients $A_{\mu\nu\lambda}^{nl}$'s. The new feature of the above step is that the Lagrangian constraint coefficients $h_{\mu\nu j}^{(1,2)}$'s also appear and can be determined in terms of the previously determined relation coefficients.

Now following a deductive proof, suppose one could have written the LCSCs and the relation coefficients of all steps, up to the nth, in terms of the HCSCs and the previously determined relation coefficients. Then we show that the same thing is also possible in the $(n+1)$th step. To show this, suppose that we have established the relation

$$\chi_\mu^{(s)} = \mathrm{FL}^* \, \Phi_\mu^{(s)} + \sum_{t=1}^{s-1} g_{\mu\nu}^{(s,t)} \chi_\nu^{(t)} \qquad s = 2, \ldots, n \tag{40}$$

between the Lagrangian and Hamiltonian constraints. This relation can also be written weakly (as derived in [3]) in the form of

$$\chi_\mu^{(s)} \stackrel{S_{s-1}}{=} \mathrm{FL}^* \, \Phi_\mu^{(s)}. \tag{41}$$

Substituting (40) in (25) gives

$$\frac{\mathrm{d}\chi_\mu^{(n)}}{\mathrm{d}t} = \mathrm{FL}^* \, \Phi_\mu^{(n+1)} + \frac{\mathrm{d}}{\mathrm{d}t} \sum_{s=1}^{n-1} g_{\mu\nu}^{(n,s)} \chi_\nu^{(s)} + v_\nu \sum_{s=1}^{n} \mathrm{FL}^* \, A_{\mu\nu\lambda}^{ns} \chi_\lambda^{(s)} - L_i \, \mathrm{FL}^* \left(\frac{\partial \Phi_\mu^{(n)}}{\partial p_i}\right) \tag{42}$$

which, after some algebra and use of (9), gives the following result:

$$\frac{\mathrm{d}\chi_\mu^{(n)}}{\mathrm{d}t} = \mathrm{FL}^* \,\Phi_\mu^{(n+1)} + \sum_{m=1}^{n-1} g_{\mu\nu}^{(n,m)} \chi_\nu^{(m+1)} + v_\nu \sum_{m\leqslant n} \mathrm{FL}^* \,A_{\mu\nu\lambda}^{nm} \left(\chi_\lambda^{(m)} - \sum_{l=1}^{m-1} g_\eta^{(m,l)} \chi_\eta^{(l)} \right) + L_i \,\mathrm{FL}^* \frac{\partial \Phi_\mu^{(n)}}{\partial p_i} + \sum_{m=1}^{n-1} g_{\mu\nu}^{(n,m)} b_{\nu i}^{(m)} L_i + \sum_{m=1}^{n-1} \left[\frac{\partial g_{\mu\nu}^{(n,m)}}{\partial \dot{q}_j} \chi_\nu^{(m)} + g_{\mu\nu}^{(n,m)} \sum_{l=1}^{m-1} h_{\nu\eta j}^{(m,l)} \chi_\eta^{(l)} \right] \ddot{q}_j. \tag{43}$$

Comparing this relation with the consistency condition of the Lagrangian constraint $\chi_\mu^{(n)}$ (relation (9) for $s = n$), we can read the nth level LCSCs as follows:

$$b_{\mu i}^{(n)} = \mathrm{FL}^* \frac{\partial \Phi_\mu^{(n)}}{\partial p_i} + \sum_{m=1}^{n-1} g_{\mu\nu}^{(n,m)} b_{\nu i}^{(m)} \tag{44}$$

$$h_{\mu\nu j}^{(s,t)} = \frac{\partial g_{\mu\nu}^{(s,t)}}{\partial \dot{q}_j} + \sum_{l=t+1}^{s-1} g_{\mu\eta}^{(s,l)} h_{\eta\nu j}^{(l,t)}. \tag{45}$$

The following relation between the Lagrangian and Hamiltonian constraints also results from (43):

$$\chi_\mu^{(n+1)} = \mathrm{FL}^* \,\Phi_\mu^{(n+1)} + \sum_{m=1}^{n-1} g_{\mu\nu}^{(n,m)} \chi_\nu^{(m+1)} + v_\nu \sum_{m\leqslant n} \mathrm{FL}^* \,A_{\mu\nu\lambda}^{nm} \left(\chi_\lambda^{(m)} - \sum_{l=0}^{m-1} g_\eta^{(m,l)} \chi_\eta^{(l)} \right)$$

where by comparison with (40) for $s = n+1$, we can introduce the $(n+1)$th-level relation coefficients in the following way:

$$g_{\mu\nu}^{(s+1,t+1)} = g_{\mu\nu}^{(s,t)} + v_\eta \left(\mathrm{FL}^* \,A_{\mu\eta\nu}^{st+1} - \sum_{l=t+2}^{s} \mathrm{FL}^* \,A_{\mu\eta\lambda}^{sl} g_{\lambda\nu}^{(l,t+1)} \right). \tag{46}$$

As we see, in a deductive way one can calculate the Lagrangian constraint structure coefficients $b_{\mu i}^{(s)}$ and $h_{\mu\nu j}^{(s,t)}$ as well as the relation coefficients $g_{\mu\nu}^{(s,t)}$, from the Hamiltonian constraint coefficients and Lagrange multipliers $v_\mu(q,\dot{q})$. The story goes on till the last step $n = k$. For the last step we should replace (25) with another (not very different) formula derived by acting on (24) with the operator FL*. The only difference is that the left-hand side of (25) is replaced by $\mathrm{FL}^* \sum_{s=1}^{k} B_{\mu\nu}^{(s)} \Phi_\nu^{(s)}$. Following the same procedure leading to relation (43) and using (40), we finally have:

$$\frac{\mathrm{d}\chi_\mu^{(k)}}{\mathrm{d}t} = \sum_{s=1}^{k} \mathrm{FL}^* \,B_{\mu\nu}^{(s)} \left(\chi_\nu^{(s)} - \sum_{t=1}^{s-1} g_{\nu\lambda}^{(s,t)} \chi_\lambda^{(t)} \right) + \sum_{m=1}^{k-1} g_{\mu\nu}^{(k,m)} \chi_\nu^{(m+1)} + v_\nu \sum_{m\leqslant k} \mathrm{FL}^* \,A_{\mu\nu\lambda}^{km} \left(\chi_\lambda^{(m)} - \sum_{l=0}^{m-1} g_\eta^{(m,l)} \chi_\eta^{(l)} \right) + L_i \,\mathrm{FL}^* \frac{\partial \Phi_\mu^{(k)}}{\partial p_i} + \sum_{m=1}^{k-1} g_{\mu\nu}^{(k,m)} b_{\nu i}^{(m)} L_i + \sum_{m=1}^{k-1} \left[\frac{\partial g_{\mu\nu}^{(k,m)}}{\partial \dot{q}_j} \chi_\nu^{(m)} + g_{\mu\nu}^{(k,m)} \sum_{l=1}^{m-1} h_{\mu\eta j}^{(m,l)} \chi_\eta^{(l)} \right] \ddot{q}_j. \tag{47}$$

Comparing this result with the last step consistency condition of Lagrangian constraints (relation (11)), we can read the Lagrangian coefficients $a_{\mu\nu}^{(s)}$ (besides the coefficients $b_{\mu i}^{(k)}$ and $h_{\mu\nu j}^{(k,t)}$ which have the same form as (44) for $n = k$) as follows:

$$a_{\mu\nu}^{(s)} = g_{\mu\nu}^{(k,s-1)} + \mathrm{FL}^* B_{\mu\nu}^{(s)} + v_\eta \,\mathrm{FL}^* A_{\mu\eta\nu}^{ks} - \sum_{l=s+1}^{k} (\mathrm{FL}^* B_{\mu\lambda}^{(l)} + v_\eta \,\mathrm{FL}^* A_{\mu\eta\lambda}^{kl}) g_{\lambda\nu}^{(l,s)}. \qquad (48)$$

This completes our main goal of determining all the LCSCs and relation coefficients in terms of HCSCs.

4. Example

Consider the Lagrangian [10]

$$L = \tfrac{1}{2}[(\dot{q}_2 - e^{q_1})^2 + (\dot{q}_3 - q_2)^2]. \qquad (49)$$

There is only one primary Lagrangian constraint,

$$\chi^{(1)} = e^{q_1}(e^{q_1} - \dot{q}_2) = -L_1. \qquad (50)$$

We have only one constraint chain, i.e. $m = 1$ (see equation (6)), and one can suppress the index μ throughout in what follows. Using the algorithm of subsection 2.1, the consistency of $\chi^{(1)}$ leads to

$$\frac{\mathrm{d}\chi^{(1)}}{\mathrm{d}t} = \chi^{(2)} - e^{q_1} L_2 \qquad (51)$$

where

$$\chi^{(2)} = e^{q_1}(\dot{q}_3 - q_2) + \dot{q}_1 \chi^{(1)}. \qquad (52)$$

Choosing $\chi^{(2)}$ as above, the only *non-vanishing* $b_i^{(1)}$ is

$$b_2^{(1)} = -e^{q_1}. \qquad (53)$$

The consistency condition for $\chi_\mu^{(2)}$ completes the chain, as one can see

$$\frac{\mathrm{d}\chi^{(2)}}{\mathrm{d}t} = -\dot{q}_1^2 \chi^{(1)} + 2\dot{q}_1 \chi^{(2)} + [-\dot{q}_1^2 L_1 - \dot{q}_1 e^{q_1} L_2 + e^{q_1} L_3] + [\chi^{(1)} \ddot{q}_1]. \qquad (54)$$

Again comparing with (11) gives the other LCSCs as follows:

$$\begin{array}{lll} b_1^{(2)} = -\dot{q}_1^2 & b_2^{(2)} = -\dot{q}_1 e^{q_1} & b_3^{(2)} = e^{q_1} \\ a^{(1)} = -\dot{q}_1^2 & a^{(2)} = 2\dot{q}_1 & \\ h_1^{(2,1)} = 1. & & \end{array} \qquad (55)$$

Now let us proceed to the Hamiltonian formulation. It is obvious from the Lagrangian (49) that $\Phi^{(0)} = p_1$ is the primary constraint. The remaining canonical momenta from (12) are

$$\begin{aligned} p_2 &= \mathcal{P}_2(q, \dot{q}) = \dot{q}_2 - e^{q_1} \\ p_3 &= \mathcal{P}_3(q, \dot{q}) = \dot{q}_3 - q_2 \end{aligned} \qquad (56)$$

and the canonical Hamiltonian can be written as

$$H = \tfrac{1}{2}(p_2^2 + p_3^2) + p_2 e^{q_1} + q_2 p_3. \qquad (57)$$

From (22) with the Hamiltonian (57), the constraint chain is

$$\begin{aligned}\Phi^{(0)} &= p_1\\ \Phi^{(1)} &= -p_2 e^{q_1}\\ \Phi^{(2)} &= p_3 e^{q_1}.\end{aligned} \tag{58}$$

All of the constraints (58) are first class (as expected), and the *non-vanishing* coefficients A^{nl} (see equation (23)) are

$$A^{11} = -1 \qquad A^{22} = 1. \tag{59}$$

The constraint $\Phi^{(2)}$ is at the end of the constraint chain and its Poisson bracket with Hamiltonian (57) strongly vanishes, i.e. comparing with (24) we have

$$B^{(l)} = 0 \qquad l = 0, 1, 2. \tag{60}$$

Next, we want to examine the validity of the relationship between the LCSCs and HCSCs, as investigated in section 3.

First using (56) and (58) one can easily see that

$$\mathrm{FL}^* \, \Phi^{(1)} = -\mathcal{P}_2 e^{q_1} = \chi^{(1)} \tag{61}$$

in agreement with (31). Following the same lines through relations (32)–(35), we can write

$$\chi^{(2)} = \mathrm{FL}^* \, \Phi^{(2)} + \dot{q}_1 \chi^{(1)} \tag{62}$$

which, by noting (59), verifies the validity of (34) with the identification

$$g^{(2,1)}(q, \dot{q}) = v(q, \dot{q}) = \dot{q}_1. \tag{63}$$

At this step, the relations (33) are also fulfilled as follows:

$$\begin{aligned}\mathrm{FL}^* \, \frac{\partial \Phi^{(1)}}{\partial p_1} &= 0 = b_1^{(1)}\\ \mathrm{FL}^* \, \frac{\partial \Phi^{(1)}}{\partial p_2} &= -e^{q_1} = b_2^{(1)}\\ \mathrm{FL}^* \, \frac{\partial \Phi^{(1)}}{\partial p_3} &= 0 = b_3^{(1)}.\end{aligned} \tag{64}$$

Since $\chi^{(2)}$ and $\Phi^{(2)}$ both stand at the end of the corresponding constraint chain, relation (54) should coincide with relation (47) of section 3. The only thing that should be tested is the validity of relations (44) for $b_i^{(2)}$, (45) for $h_j^{(2,1)}$ and (48) for $a^{(l)}$. This is done as follows:

$$\begin{aligned}b_1^{(2)} &= \mathrm{FL}^* \, \frac{\partial \Phi^{(2)}}{\partial p_1} + g^{(2,1)} b_1^{(1)} = -\dot{q}_1^2\\ b_2^{(2)} &= \mathrm{FL}^* \, \frac{\partial \Phi^{(2)}}{\partial p_2} + g^{(2,1)} b_2^{(1)} = -\dot{q}_1 e^{q_1}\\ b_3^{(2)} &= \mathrm{FL}^* \, \frac{\partial \Phi^{(2)}}{\partial p_3} + g^{(2,1)} b_3^{(1)} = e^{q_1}\\ h^{(2,1)} &= \frac{\partial g^{(2,1)}}{\partial \dot{q}_1} = 1\\ a^{(1)} &= v \, \mathrm{FL}^* \, A^{21} - \sum_{l=2}^{2} v \, \mathrm{FL}^* \, A^{2l} g^{(l,1)} = -\dot{q}_1^2\\ a^{(2)} &= g^{(2,1)} + v \, \mathrm{FL}^* \, A^{22} = 2\dot{q}_1\end{aligned} \tag{65}$$

where all are in agreement with (55), as expected.

Acknowledgments

The authors thank Isfahan University of Technology for financial support. They also thank V Karimipur and M Khorami for reading the manuscripts and the second referee for useful comments.

References

[1] Dirac P A M 1964 *Lectures on Quantum Mechanics* (New York: Yeshiva University Press)
[2] Sudarshan E C G and Mukunda N 1974 *Classical Dynamics, A Modern Perspective* (New York: Wiley–Interscience)
[3] Batlle C, Gomis J, Pons J M and Roman-Roy N 1986 *J. Math. Phys.* **27** 2953
[4] Pons J M 1988 *J. Phys. A: Math. Gen.* **21** 2705
[5] Gomis J, Henneaux M and Pons J M 1990 *Class. Quantum Grav.* **7** 1089
[6] Cabo A, Chaichian M and Louis Martinez D 1993 *J. Math. Phys.* **34** 5646
[7] Chaichian M and Louis Martinez D 1994 *J. Math. Phys.* **35** 6536
[8] Shirzad A 1998 *J. Phys. A: Math. Gen.* **31** 2747
[9] Henneaux M and Teitelboim C 1992 *Quantization of Gauge Systems* (Princeton, NJ: Princeton University Press)
[10] Henneaux M, Teitelboim C and Zanelli J 1990 *Nucl. Phys.* B **332** 169